Selfduality of non-linear electrodynamics with derivative corrections

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ABSTRACT: In this paper we investigate how electromagnetic duality survives derivative corrections to classical non-linear electrodynamics. In particular, we establish that electromagnetic selfduality is satisfied to all orders in α' for the four-point function sector of the four dimensional open string effective action.

Contents

1. Introduction

The symmetry between electric and magnetic fields is a fundamental property of Maxwell theory, and of certain extensions such as the nonlinear electrodynamics of Born and Infeld [1]. Consider the free Maxwell equations in $d = 4$ flat space: [1]

$$\partial_a F^{ab} = 0 \qquad \partial_a \tilde{F}^{ab} = 0 \,, \tag{1.1}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ and $\widetilde{F}$ is the Hodge dual of F, i.e, $\tilde{F}_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}$. Indeed, (1.1) is invariant under the Hodge duality transformation.

One can pose the question whether a generalization of duality invariance continues to hold for deformations of the Maxwell action. In particular, one might consider Lagrangian densities depending only on abelian field and a deformation parameter, say α', which coincides with the Maxwell Lagrangian for $\alpha' = 0$. The general Lagrangian satisfying such restrictions and leading to electromagnetic duality invariance involves an arbitrary real function of one real argument [2–4]. A particular example is the Born-Infeld Lagrangian [1]:

$$\mathcal{L} = 1 - \sqrt{-\det(\eta_{ab} + \alpha' F_{ab})} \,. \tag{1.2}$$

The duality invariance of Born-Infeld theory was established in [5].

Our purpose in this paper is to extend the deformations of the Maxwell theory to also include derivatives of the field strength F, and to investigate if the duality invariance can be preserved. This is relevant for the application in string theory, where it is known that the open string string effective action, which for slowly varying fields coincides with the Born-Infeld action, also contains derivative corrections. We establish in this paper that the

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[1] Throughout the paper, we are working in d=4, and we adopt the following convention: Minkowski metric with $diag(-, +, +, +)$ signature, where $\widetilde{\widetilde{F}} = -F$, and often we use the notation tr $FG = -F_{ab}G^{ab}$.

effective action for the open string 4-point function, truncated to four dimensions, satisfies the property of electromagnetic duality also when derivative corrections are included.

In section 2 we will briefly review basic definitions and results concerning electromagnetic duality. The duality of the terms related to the 4-point function will be discussed in section 3. In section 4 we discuss the extension to higher-order contributions to the string effective action.

2. Electromagnetic Selfduality

In this section, we briefly review some definitions and results, see also [2–4, 6] and references therein. We consider the Lagrangian to be a function $\mathcal{L}$ of one variable F. The field equation and Bianchi identity are

$$\partial_a G^{ab} = 0 \,, \qquad \partial_a \tilde{F}^{ab} = 0 \,, \tag{2.1}$$

where G is an anti-symmetric tensor of rank two defined by

$$G^{ab} = -\frac{\partial \mathcal{L}(F)}{\partial F_{ab}} \,. \tag{2.2}$$

Consider transformations which send the pair (G, F) to (G', F'):

$$\begin{pmatrix} G'(F') \\ \tilde{F}' \end{pmatrix} = D \begin{pmatrix} G(F) \\ \tilde{F} \end{pmatrix} \,, \tag{2.3}$$

with $D \in \mathrm{GL}(2, \mathbb{R})$. We can solve (2.3) for (G, F) in terms of (G', F') and then substitute in equations (2.1)-(2.2) to find the transformed version of the field equations and Lagrangian. We will assume the existence of the transformed Lagrangian $\mathcal{L}'(F')$, which satisfies

$$G'(F')^{ab} = -\frac{\partial \mathcal{L}'(F')}{\partial F'_{ab}} \,. \tag{2.4}$$

Then the property of electromagnetic duality invariance or selfduality is defined by

$$\mathcal{L}'(F) = \mathcal{L}(F) \,. \tag{2.5}$$

From (2.5) and the form of the duality transformations one derives that the duality symmetry is SO(2), that the constraint

$$\mathrm{tr}\, G\tilde{G} = \mathrm{tr}\, F\tilde{F} \tag{2.6}$$

must hold, and that the combination

$$\mathcal{L}(F) - \frac{1}{4} \mathrm{tr}\, FG \tag{2.7}$$

is duality invariant.

It is natural to ask what happens if the action also depends on derivatives of the field strength. At first sight, it seems that the analysis presented is not applicable anymore, and

hence should be modified. This also happens in the extension which includes additional scalars, e.g, axion and dilaton fields [2, 7]. In that case the duality symmetry is modified and becomes $\mathrm{SL}(2,\mathbb{R})$. In the case of derivative corrections however, most of the discussion above can be taken over if one works with the action rather than with the Lagrangian density, and uses functional differentiation [4]. We then define[2]

$$G^{ab} = -\frac{\delta}{\delta F^{ab}} S[F] \,. \tag{2.9}$$

The duality transformations retain the same form, we now assume that G' follows from an action $S'[F']$, the duality condition then reads

$$S'[F] = S[F] \,, \tag{2.10}$$

while the constraint (2.6) becomes:

$$\int d^4 x \, \mathrm{tr}\, G\tilde{G} = \int d^4 x \, \mathrm{tr}\, F\tilde{F} \,. \tag{2.11}$$

In appendix A we outline a proof of (2.11), and discuss the effect of field redefinitions. If we have an action S_0 satisfying the condition of selfduality, then of course any action related to that action by a field redefinition should also be considered to be electromagnetically selfdual. In appendix A we show that this implies that we should allow (2.11) to hold up to terms containing $\partial_a G_0{}^{ab}$, the equation of motion of S_0.

3. Selfduality of the 4-point function derivative corrections

In this section we will extend the electromagnetic selfduality of the open superstring effective action to include derivative corrections. The terms we will consider have the generic form

$$\mathcal{L}_{(m,n)} = \alpha'^m \, \partial^n F^p \,, \qquad \text{for } p = m + 2 - n/2 \,, \tag{3.1}$$

$$\mathcal{L}_m = \alpha'^m F^{m+2}, \qquad \text{for } n = 0 \,. \tag{3.2}$$

The absence of corrections with $n = 2$ has been established in [8]. All corrections for $n = 4$ have been constructed by Wyllard [9], while it is known that terms with p odd are absent. The terms with $p = 4$ have been obtained to all orders in α' [10]. We will establish in this section the electromagnetic duality of the $p = 4$ terms.

Of course electromagnetic selfduality as discussed in this paper holds in spacetime dimension $d = 4$, while the superstring corrections are obtained in $d = 10$. We will discuss electromagnetic duality of the contributions of the form (3.1), setting all other ten-dimensional fields to zero, and truncating the result to $d = 4$, i.e., by restricting the Lorentz index values to $d = 4$. Furthermore, the result can hold only order-by-order in α', in the

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[2]We use the chain rule and

$$\frac{\delta}{\delta F_{ab}(x)} F_{cd}(y) = 2\delta_{cd}^{ab} \, \delta(y - x) \,. \tag{2.8}$$

sense that for each order of α' the corresponding $p = 4$ contribution to the effective action satisfies, together with the $m = 0$ Maxwell term, electromagnetic selfduality to order m in α'.

Let us start the discussion with the $m = 4$ terms. Then the four-derivative terms are

$$\mathcal{L}_{(4,4)} = a_{(4,4)}\alpha'^4 t_8{}^{abcdefgh}\partial_k F_{ab}\partial^k F_{cd}\partial^l F_{ef}\partial_l F_{gh}\,, \tag{3.3}$$

where $t_8{}^{abcdefgh}$ is antisymmetric in the pairs ab, cd, etc., and is symmetric under the exchange of such pairs. It expands as follows for arbitrary antisymmetric matrices M_i:

$$t_8{}^{abcdefgh}M_{1\,ab}M_{2\,cd}M_{3\,ef}M_{4\,gh} = 8\left(\mathrm{tr}M_1M_2M_3M_4 + \mathrm{tr}M_1M_3M_2M_4 + \mathrm{tr}M_1M_3M_4M_2\right)$$
$$-2\left(\mathrm{tr}M_1M_2\mathrm{tr}M_3M_4 + \mathrm{tr}M_1M_3\mathrm{tr}M_2M_4 + \mathrm{tr}M_1M_4\mathrm{tr}M_2M_3\right). \tag{3.4}$$

The values of the constants $a_{m,2m-4}$ can be found in [10]. The equation of motion of the combination $\mathcal{L}_0 + \mathcal{L}_{(4,4)}$ contains

$$G^{ab} = F^{ab} + G^{ab}_{(4,4)}\,, \quad G^{ab}_{(4,4)} = 4a_{(4,4)}\alpha'^4 t_8{}^{abcdefgh}\partial_k\left(\partial^k F_{cd}\partial^l F_{ef}\partial_l F_{gh}\right). \tag{3.5}$$

To establish electromagnetic selfduality we have to establish that (2.11) holds. It only makes sense to verify this to order α'^4, since in higher orders other contributions to the effective action would interfere. Since the α'^0 terms in (2.11) cancel we have to verify that

$$I = \int d^4x\,\mathrm{tr}\,\widetilde{F}G_{(4,4)} = 0\,. \tag{3.6}$$

After partial integration (3.6) takes on the form

$$I = \int d^4x\,t_8{}^{abcdefgh}\partial_k\widetilde{F}_{ab}\partial^k F_{cd}\partial_l F_{ef}\partial^l F_{gh} = 0. \tag{3.7}$$

The crucial property, which in fact holds to all orders in α', is that in $\mathcal{L}_{(m,2m-4)}$ the indices of the fieldstrengths F are all contracted amongst each other, and therefore also the derivatives are contracted [10]. The identity $(F_k \equiv \partial_k F)$

$$(\widetilde{F}_k F_l + \widetilde{F}_l F_k)_a{}^b = -\tfrac{1}{2}\delta^b_a\,\mathrm{tr}\,\widetilde{F}_k F_l\,, \tag{3.8}$$

in combination with the complete symmetry of t_8, can then be used to express the traces over four matrices resulting from the expansion of (3.7) in terms of products of traces over two matrices. This leads to the required cancellation.

For higher orders in α' the $p = 4$ terms contain more derivatives, but again these are all contracted with each other, while the tensor structure of the fieldstrengths remains the same. Essentially one has to show that

$$t_8{}^{abcdefgh}\left[\widetilde{F}_{1\,ab}F_{2\,cd}F_{3\,ef}F_{4\,gh} + F_{1\,ab}\widetilde{F}_{2cd}F_{3ef}F_{4\,gh} + F_{1\,ab}F_{2\,cd}\widetilde{F}_{3\,ef}F_{4\,gh} + F_{1\,ab}F_{2\,cd}F_{3\,ef}\widetilde{F}_{4\,gh}\right], \tag{3.9}$$

where the subscripts $1, 2, 3, 4$ indicate the derivative structure, vanishes. Using again (3.8) and the symmetry of t_8 one establishes that (3.9) vanishes independently of the precise way the derivatives are contracted.

This gives the desired result: electromagnetic duality survives, to this order in α', the addition of derivative corrections. Note that in verifying (2.11) to order α'^4 there are no term proportional to $\partial_a G_0{}^{ab}$ left over. Had we started from the $p = 4$ terms in a different basis, for instance that given in [9] for $m = 4$, then indeed (2.11) would hold only up to terms that vanish on-shell.

4. Conclusions

It would be of interest to use electromagnetic selfduality to constrain, or to determine, the derivative corrections to the Born-Infeld action that are not known explicitly. However, it is well-known that already the Born-Infeld action itself is not the only selfdual deformation of the Maxwell action, the ambiguity can be parametrized by a real function of one variable [2]. From the previous section it clear that $\mathcal{L}_{(m,2m-4)}$ is not the only $p = 4$ action with derivative corrections that satisfies (2.11) to order α'^4. Indeed, we found that the result depends only on the presence of the tensor t_8 and on the fact that there are no contractions between derivatives and fieldstrengths. The result is independent of the precise way the derivatives are placed.

Given these ambiguities, it is clear that electromagnetic duality can only constrain but not determine the derivative corrections to the terms related to the six-point function, $p = 6$. For the four-derivative terms $n = 4$ we do have the result of [9]. The method used in section 3 is however not applicable, because the property of having no contractions between field strengths and derivatives no longer holds. Nevertheless, it would be interesting to extend the analysis of selfduality to those terms.

Another extension would be to add derivative corrections to the $SL(2, \mathbb{R})$ invariant extension of Born-Infeld [7]. This problem is currently under investigation [11].

Acknowledgments

A. Integral Form of the selfduality condition and field redefinitions

We derive the consistency condition (2.11) for infinitesimal duality transformations. Then

$$G'^{ab}[F'] = G^{ab}[F] + \lambda \tilde{F}^{ab}, \quad \tilde{F}'^{ab} = \tilde{F}^{ab} - \lambda G^{ab}[F], \tag{A.1}$$

with

$$G'^{ab}[F'] = -\frac{\delta S'[F']}{\delta F'_{ab}(x)}. \tag{A.2}$$

Selfduality (2.10) implies

$$G'^{ab}[F', x] = -\frac{\delta S[F']}{\delta F'_{ab}(x)} = -\left(\frac{\delta S[F]}{\delta F'_{ab}(x)} + \frac{\delta}{\delta F_{ab}(x)}\delta S[F]\right), \tag{A.3}$$

where we use

$$\delta S[F] = S[F'] - S[F] \tag{A.4}$$

and

$$\frac{\delta}{\delta F'_{ab}(x)} \delta S[F] = \frac{\delta}{\delta F_{ab}(x)} \delta S[F] + \mathcal{O}(\lambda^2) \, . \tag{A.5}$$

$\delta S[F]/\delta F'_{ab}$ can be evaluated as follows:

$$\begin{aligned}
\frac{\delta S[F]}{\delta F'_{ab}(x)} &= \tfrac{1}{2} \int d^4 y \, \frac{\delta S[F]}{\delta F_{cd}(y)} \frac{\delta F_{cd}(y)}{\delta F'_{ab}(x)} \\
&= \tfrac{1}{2} \int d^4 y \, \frac{\delta S[F]}{\delta F_{cd}(y)} \frac{\delta}{\delta F'_{ab}(x)} \Big(F'_{cd}(y) - \lambda \tilde{G}_{cd}[F, y] \Big) \\
&= -G^{ab}[F, x] + \tfrac{\lambda}{2} \int d^4 y \, G^{cd}[F, y] \frac{\delta}{F_{ab}(x)} \tilde{G}_{cd}[F, y] \\
&= -G^{ab}[F, x] + \tfrac{\lambda}{4} \frac{\delta}{F_{ab}(x)} \Big(\int d^4 y G^{cd}[F, y] \tilde{G}_{cd}[F, y] \Big) \, .
\end{aligned} \tag{A.6}$$

Substituting (A.6) in (A.3) yields

$$G'^{ab}[F', x] = G^{ab}[F, x] - \frac{\delta}{F_{ab}(x)} \Big(\delta S[F] + \tfrac{\lambda}{4} \int d^4 y G^{cd}[F, y] \tilde{G}_{cd}[F, y] \Big) \, . \tag{A.7}$$

On the other hand, from the variation (A.1) of G it follows

$$G'^{ab}[F', x] = G^{ab}[F, x] - \frac{\delta}{F_{ab}(x)} \Big(-\tfrac{\lambda}{4} \int d^4 y F^{cd}(y) \tilde{F}_{cd}(y) \Big) \, . \tag{A.8}$$

The variation of S is

$$\delta S[F] = \tfrac{1}{2} \int d^4 y \, \frac{\delta S[F]}{\delta F_{cd}(y)} \delta F_{cd}(y) = -\tfrac{\lambda}{2} \int d^4 y \, G^{cd}[F, y] \tilde{G}_{cd}[F, y] \tag{A.9}$$

Inserting (A.9) into (A.7) and comparing the resulting expression to (A.8), one finds the integrated form selfduality condition (2.11).

If we have a selfdual action S_0 satisfying (2.11), and we perform a field redefinition on the vector potential, (2.11) will only be satisfied modulo terms proportional to $\partial_a G_{0ab}$. To see this, write the new action as

$$S = S_0 + S_1 \, , \tag{A.10}$$

where S_1 is of the form

$$S_1 = \int d^4 x \, V_b[F, x] \partial_a G_0{}^{ab} \, . \tag{A.11}$$

Then

$$G^{ab}(x) = G_0{}^{ab}(x) - \int d^4 y \left(\frac{\delta V_d(y)}{\delta F_{ab}(x)} \partial_c G_0{}^{cd}(y) - \partial_c V_d(y) \frac{\delta G_0{}^{cd}(y)}{\delta F_{ab}(x)} \right) \, . \tag{A.12}$$

Using the fact that S_0 satisfies (2.11) the remaining condition for the selfduality of S is

$$0 = \int d^4 x d^4 y \left(\tilde{G}_{0ab}(x) \frac{\delta V_d(y)}{\delta F_{ab}(x)} \partial_c G_0{}^{cd}(y) - \partial_c V_d(y) \frac{\delta G_0{}^{cd}(y)}{\delta F_{ab}(x)} \tilde{G}_{0ab}(x) \right) \, . \tag{A.13}$$

The second term in (A.13) vanishes. This can be seen by using

$$\frac{\delta G_0{}^{cd}(y)}{\delta F_{ab}(x)} = \frac{\delta G_0{}^{ab}(x)}{\delta F_{cd}(y)} \tag{A.14}$$

and (2.11) for S_0, the result then contains $\partial_a \widetilde{F}^{ab}$ which vanishes. The remaining term is proportional to $\partial_c G_0{}^{cd}(y)$ so that indeed we see that selfduality holds modulo the S_0 equation of motion.

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